Dissipation Bound for Thermodynamic Control

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Biological and engineered systems operate by coupling function to the transfer of heat and/or particles down a thermal or chemical gradient. In idealized *deterministically* driven systems, thermodynamic control can be exerted reversibly, with no entropy production, as long as the rate of the protocol is made slow compared to the equilibration time of the system. Here we consider *fully realizable*, entropically driven systems where the control parameters themselves obey rules that are reversible and that acquire directionality in time solely through dissipation. We show that when such a system moves in a directed way through thermodynamic space, it must produce entropy that is on average larger than its generalized displacement as measured by the Fisher information metric. This distance measure is subextensive but cannot be made small by slowing the rate of the protocol.

It is generally believed that the fundamental laws of physics preserve phase space volume and are almost the same in reverse [1]. However, the macroscopic world with which we interact is dominated by biological and engineered machines that are *dissipative* and strongly break time reversibility; although their microscopic components obey the phase space volume preserving laws of physics, they couple movement to the production of entropy, leading them to move more often than not in a particular direction. The macroscopic engines ubiquitous in engineered systems are sufficiently large and dissipative as to rarely move in reverse. However, the processive but stochastic molecular motors common in biology do indeed take frequent steps backwards [2–7], reminding us of the entropic and necessarily stochastic underpinnings of their directed motion.

How much entropy must be produced to ensure that a thermodynamic system moves forward? Most analysis of this question assumes that the system is driven by changes in control parameters λ^μ (e.g., the pressure in a Carnot engine), whose dynamics are external to the problem. As these control parameters are moved *deterministically*, the system state x changes stochastically and, on average, the total entropy of the system and its coupled heat bath(s) increases. However, as a deterministic protocol is made infinitely slow, it becomes thermodynamically reversible, producing no entropy. In this adiabatic limit, a deterministic Carnot engine can run with arbitrarily small dissipation. The aim of this Letter is to present a new bound that applies to *fully realizable*, entropically driven systems, where all components obey microscopic reversibility, acquiring directionality in time only through dissipation. We show that controlling such a system entails a finite entropic cost that cannot be removed by lengthening the time of the protocol. A fully realizable Carnot engine must dissipate a finite but subextensive amount of entropy in every cycle.

The last two decades have seen enormous progress on understanding deterministically driven systems, with surprising equalities applying to ensemble averages of entropy production [8] and statistical properties of microscopic trajectories [9]. In addition, new bounds constrain the entropy production associated with finite time protocols. Using arguments from linear response, Sivak and Crooks [10] showed that any protocol starting at $\vec{\lambda}_0$ and ending at $\vec{\lambda}_f$, to be completed in time $t_{\max}$ must dissipate at least

$$\langle \Delta S_{\text{tot}} \rangle \geq \frac{\tilde{\mathcal{L}}^2(\vec{\lambda}_0, \vec{\lambda}_f)}{t_{\max}} + \mathcal{O}\left(\frac{1}{t_{\max}^2}\right) \tag{1}$$

entropy (in units where Boltzmann's constant $k_B = 1$). Here $\tilde{\mathcal{L}}(\vec{\lambda}_0, \vec{\lambda}_f)$ is the geodesic distance between initial and final control parameters in the metric space where infinitesimal length $d\tilde{l}$ between λ and $\lambda + d\lambda$ is given by $d\tilde{l}^2 = \tilde{g}_{\mu\nu}(\lambda)d\lambda^\mu d\lambda^\nu$, where here and throughout repeated indices are summed. $\tilde{g}$ is a "friction tensor" defined by

$$\tilde{g}_{\mu\nu}(\lambda) = \int_0^\infty dt \langle [\Phi_\mu(0) - \langle \Phi_\mu \rangle_\lambda][\Phi_\nu(t) - \langle \Phi_\nu \rangle_\lambda] \rangle_\lambda, \tag{2}$$

where $\Phi_\mu(x)$ is the conjugate force to λ^μ, entering the Hamiltonian through a term $-\lambda^\mu \Phi_\mu(x)$. Crooks [11] and Burbea and Rao [12] considered dissipation in related protocols in which thermodynamic parameters are moved in N discrete steps, with the system coming fully to equilibrium between each step. In this case the dissipation is bounded by

$$\langle \Delta S_{\text{tot}} \rangle \geq \frac{\mathcal{L}^2(\vec{\lambda}_0, \vec{\lambda}_f)}{n_{\text{steps}}} + \mathcal{O}\left(\frac{1}{n_{\text{steps}}^2}\right), \tag{3}$$

where the tilde-free $\mathcal{L}(\vec{\lambda}_0, \vec{\lambda}_f)$ is the geodesic distance between initial and final control parameters in the metric space defined by the Fisher information matrix (FIM). The FIM, $g_{\mu\nu} = \langle \partial_\mu \partial_\nu \log[p(x)] \rangle$ measures statistical distinguishability between nearby models [13,14] and in this parametrization takes the form

$$g_{\mu\nu}(\lambda) = \langle \Phi_\mu \Phi_\nu \rangle_\lambda - \langle \Phi_\mu \rangle_\lambda \langle \Phi_\nu \rangle_\lambda = \frac{\partial \langle \Phi_\mu \rangle_\lambda}{\partial \lambda^\nu}. \quad (4)$$

In each of these deterministically driven cases, the entropy production can be reduced to an arbitrarily small value by taking the limit where either $t_{\max} \to \infty$ or $n_{\text{steps}} \to \infty$. Here we consider fully realizable systems, where the control parameters themselves obey dynamics that obey time reversal invariance, moving forward more often than backwards only due to the increased volume of phase space associated with movement in that direction. In this entropically driven limit, there is an intrinsic cost associated with thermodynamic control, which, even as the procedure is made arbitrarily slow and fine grained remains bounded by

$$\langle \Delta S_{\text{tot}} \rangle \geq 2\mathcal{L}(\vec{\lambda}_0, \vec{\lambda}_f), \quad (5)$$

where $\mathcal{L}$ is the geodesic distance in the same metric space of the FIM defined in Eq. (4). Demonstrating this bound and discussing its implications are the subject of this Letter.

Before looking at the more general case, consider a system with particles interacting through some Hamiltonian $H_{\text{sys}}(x)$, where x denotes the microstate of the system (see Fig. 1). Particle number, $N(x)$ is a function of x and at time t the system may exchange particles with particle reservoir $i(t)$ at chemical potential λ_i/β and energy with a thermal bath at inverse temperature β. After reaching equilibrium with reservoir i, the probability of finding the controlled system in state x is given by an appropriate Boltzmann distribution,

$$P_i(x) = \frac{\exp\left[-\beta H_{\text{sys}}(x) + \lambda_i N(x)\right]}{Z_i}. \quad (6)$$

With these definitions, $\Omega_i = -\log(Z_i)/\beta$ is the grand potential, and the Helmholtz free energy is given by

$$F_{\text{sys},i} = \langle H_{\text{sys}} \rangle_i - S_{\text{sys},i}/\beta = \Omega_i + \lambda_i \langle N \rangle_i/\beta, \quad (7)$$

where $\langle N \rangle_i = \partial \log(Z_i)/\partial \lambda$. We are interested in changes to the combined entropy, $S_{\text{sys}+} = S_{\text{sys}} + S_{\text{bath}} + \sum_i S_i$, where $\Delta S_{\text{bath}} = -\beta \Delta E$ so that $\Delta S_{\text{bath}} + \Delta S_{\text{sys}} = -\beta \Delta F$ and $\Delta S_i = \lambda_i \Delta N_i$. We will also need the FIM which can be expressed as [see Eq. (4)]:

$$g_i = \langle N^2 \rangle_i - \langle N \rangle_i^2 = \frac{\partial \langle N \rangle_i}{\partial \lambda} = -\beta \frac{\partial^2 \Omega_i}{\partial \lambda^2}, \quad (8)$$
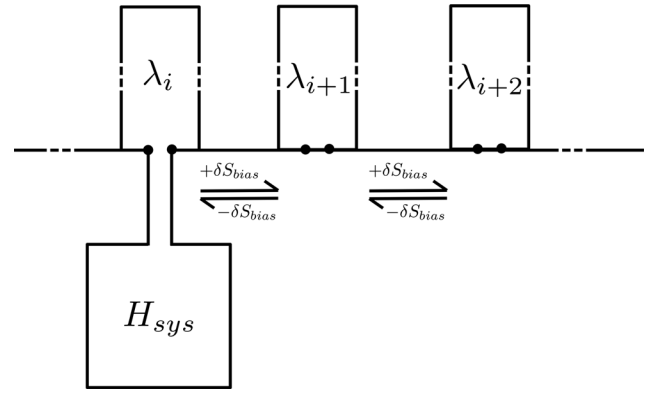

FIG. 1. A fully realizable system consisting of a controlled system of interest with Hamiltonian H_{sys}, a series of particle reservoirs i each with chemical potential λ_i/β, and an unspecified source of bias entropy δS_{bias}. The controlled system is also in contact with a thermal bath at inverse temperature β with which it can exchange energy (not shown). At a fast time scale the system exchanges particles with bath i, and at a much slower time scale it hops from bath i to $i \pm 1$, producing $\pm \delta S_{\text{bias}}$ entropy. When δS_{bias} is small, the system tends to drift, hopping both forwards and backwards, during which it moves particles down their chemical gradient, dissipating additional entropy. As argued in the text, average forward movement requires a minimum entropy production that cannot be removed either by making $d\lambda = \lambda_{i+1} - \lambda_i$ or δS_{bias} small.

where the tensor indices of g_i are omitted as the system contains only a single parameter.

In addition to dynamics in which the system comes to equilibrium with reservoir i, our system may also disconnect from reservoir i and reconnect to reservoirs $i + 1$ or $i - 1$, that are held at $\lambda_{i\pm 1} = \lambda_i \pm d\lambda$. Let us first consider dynamics in which these steps happen in the forward and reverse directions with equal rates. In this directionless steady state, the complete system is *not* in thermal equilibrium. In fact, as it diffuses back and forth, the controlled system will tend to move particles down their concentration gradients. Consider a sequence in which the system starts in equilibrium connected to reservoir i, hops to reservoir $i + 1$, comes to equilibrium there, and then hops back to reservoir i where it again comes to equilibrium. At the end of this sequence, the entropy of the thermal bath and controlled system remain unchanged. However, on average, particles have been transferred from reservoir $i + 1$ to reservoir i; during the forward hop the system carries, on average, $\langle N \rangle_i$ particles while in the backwards trajectory it carries $\langle N \rangle_{i+1} = \langle N \rangle_i + g_i d\lambda + \mathcal{O}(d\lambda)^2$ particles [see Eq. (8)].

In fact, a hop in either direction produces, on average, total entropy $g_i (d\lambda)^2/2$. Consider a sequence in which the systems hops from bath i to bath $i \pm 1$, where it comes to equilibrium. Expanding the Helmholtz free energy [Eq. (7)] in $d\lambda$, we find that $\beta \Delta F_{\text{sys}} = \pm g_i \lambda_i d\lambda + [(g_i/2) + (\lambda_i/2) \times (\partial^2 \langle N \rangle_i/\partial \lambda^2)](d\lambda)^2 + \mathcal{O}(d\lambda)^3$. The entropy of reservoir

$i \pm 1$ also changes. On average, $\langle N \rangle_{i+1} - \langle N \rangle_i$ particles flow out of it, changing its entropy by $\Delta S_{i\pm1} = (\lambda_i \pm d\lambda)[\pm g_i d\lambda + \frac{1}{2}(\partial^2 \langle N \rangle_i / \partial \lambda^2)(d\lambda)^2 + \mathcal{O}(d\lambda)^3]$. Thus, during this procedure,

$$\langle \Delta S_{\text{sys+}} \rangle_{i \to i\pm1} = -\beta \Delta F + \sum_i \Delta S_i = \frac{g_i}{2}(d\lambda)^2 + \mathcal{O}(d\lambda)^3. \tag{9}$$

So far this system has no well-defined direction of operation, despite its dissipative nature. To bias towards movement in the forward direction, a step from one particle reservoir to the next may couple to the dissipation of an unspecified source of biasing entropy δS_{bias}, so that a step from $i \to i+1$ is more likely than one in the reverse direction, with the forward and reverse rates $r_\pm$, as well as the average number of forward and reverse hops, $n_\pm$ related by [9,15]:

$$\langle n_- \rangle = \langle n_+ \rangle \exp(-\delta S_{\text{bias}}). \tag{10}$$

Microscopically, δS_{bias} could arise from many sources, provided they produce entropy. For example, forward movement might be accompanied by the transfer of a fixed amount of energy down a thermal gradient as in an engine, or of particles down a chemical gradient. It is important to note that the reverse step does indeed reduce the entropy of the world by δS_{bias}.

Let us now consider the average dissipation associated with moving system parameters from λ_0 to λ_f, over a trajectory through which $g_i = g$ is constant so that $\mathcal{L}(\lambda_0, \lambda_f) = \sqrt{g}|\lambda_f - \lambda_0|$. The total average dissipation associated with this trajectory is given by the sum of bias related dissipations $S_{\text{bias}} = (n_+ - n_-)\delta S_{\text{bias}}$ and dissipations associated with hops, $\Delta S_{\text{sys+}}$:

$$\langle \Delta S_{\text{tot}} \rangle = \langle n_+ - n_- \rangle \delta S_{\text{bias}} + \langle n_+ + n_- \rangle \langle \Delta S_{\text{sys+}} \rangle. \tag{11}$$

In any trajectory which moves from λ_0 to λ_f, $n_+ - n_- = (\lambda_f - \lambda_0)/d\lambda$. Combining with Eqs. (9) and (10) the total average dissipation is given by

$$\langle \Delta S_{\text{tot}} \rangle = \mathcal{L}(\lambda_0, \lambda_f)\left(\frac{\delta S_{\text{bias}}}{\sqrt{g(d\lambda)^2}} + \frac{\sqrt{g(d\lambda)^2}}{2\tanh(\delta S_{\text{bias}}/2)}\right). \tag{12}$$

An idealized system could have any stepsize $d\lambda$ and any δS_{bias}. Optimizing over these to find the values that minimize the dissipation we find that

$$g(d\lambda^{\text{opt}})^2 = (\delta S_{\text{bias}}^{\text{opt}})^2, \delta S_{\text{bias}}^{\text{opt}} \to 0, \langle \Delta S_{\text{tot}}^{\text{opt}} \rangle = 2\mathcal{L}(\lambda_0, \lambda_f), \tag{13}$$

demonstrating Eq. (5) for this system.

To achieve the bound, the system must be in the continuum limit where $d\lambda$ and δS_{bias} are small. Furthermore, if $\delta S_{\text{bias}} > \sqrt{g}d\lambda$ then entropy is produced primarily in breaking the time-reversal invariance of the control parameters, though the work done on the system can be made arbitrarily close to the change in free energy. On the other hand, if $\delta S_{\text{bias}} < \sqrt{g}d\lambda$ then most entropy is produced in futile cycles in which the system moves back and forth in λ space. Insight into the optimum parameters can be gained by considering the "proper" velocity and diffusion coefficient where proper distance is measured by the FIM (where for generality we have added back indices on g):

$$V = \sqrt{g_{\mu\nu} \frac{\partial \langle \lambda^\mu \rangle}{\partial t} \frac{\partial \langle \lambda^\nu \rangle}{\partial t}},$$

$$D = \frac{1}{2} g_{\mu\nu} \frac{\partial [\lambda^\mu \lambda^\nu]}{\partial t}, \tag{14}$$

where square brackets denote a second cumulant. It is important to note that these each have dimensions of inverse time since proper distance as measured by the FIM is dimensionless. For this system, in the continuum limit, $V = \sqrt{g}r_+ d\lambda \delta S_{\text{bias}}$, while $D = gr_+(d\lambda)^2$. As such, the optimum occurs when $D = V$.

These results also hold for a generalized system with multiple control parameters λ^μ, which could be baths of different particles, temperature, displacements, or other quantities conjugate to a generalized force. Taking a continuum trajectory, we assume that the controlled system will be moved through a one-dimensional track $\vec{\lambda}(\tau)$, with motion along the track consisting of (one-dimensional) diffusion and drift, microscopically arising from dynamics analogous to those considered in the one dimensional example. At a particular point $\vec{\lambda}(\tau)$, using the definitions provided in Eq. (14), $V = (d\langle \tau \rangle/dt)\sqrt{g_{\mu\nu}(\partial \lambda^\mu/\partial \tau)(\partial \lambda^\nu/\partial \tau)}$ and $D = \frac{1}{2}(d[\tau^2]/dt) \times [g_{\mu\nu}(\partial \lambda^\mu/\partial \tau)(\partial \lambda^\nu/\partial \tau)]$. Similar analysis shows that in this multiparameter continuum system, entropy is produced at the following rate:

$$\frac{d\langle S_{\text{sys+}} \rangle}{dt} = D, \frac{d\langle S_{\text{bias}} \rangle}{dt} = V^2/D, \tag{15}$$

where S_{bias} and $S_{\text{sys+}}$ are defined as before. The average amount of time it takes to traverse a segment of length $d\tau$ is given by $\langle dt \rangle = \sqrt{g_{\mu\nu}(\partial \lambda^\mu/\partial \tau)(\partial \lambda^\nu/\partial \tau)}d\tau/V$, so that the dissipation associated with movement along a segment of length $d\tau$ is given by $\langle dS_{\text{tot}} \rangle = (D/V + V/D)\sqrt{g_{\mu\nu}(\partial \lambda^\mu/\partial \tau)(\partial \lambda^\nu/\partial \tau)}d\tau$. This is minimized when $D^{\text{opt}} = V^{\text{opt}}$ so that the minimum average dissipation associated with a stochastic protocol that on average moves from $\vec{\lambda}_0$ to $\vec{\lambda}_f$ along path $\vec{\lambda}(\tau)$ is given by

$$\langle \Delta S_{\text{tot}}^{\text{opt}} \rangle = 2 \int_{\tau_0}^{\tau_f} \sqrt{g_{\mu\nu} \frac{\partial \lambda^\mu}{\partial \tau} \frac{\partial \lambda^\nu}{\partial \tau}} d\tau \geq 2\mathcal{L}(\vec{\lambda}_0, \vec{\lambda}_f). \quad (16)$$

Note that the first equality defines (twice) the path length. Thus, the inequality below follows from the definition of $\mathcal{L}$ as the length of the minimum length (geodesic) path. Equality is achieved only when $\vec{\lambda}(\tau)$ is the minimum length geodesic connecting $\vec{\lambda}_0$ with $\vec{\lambda}_f$.

It is interesting to compare the bound presented here with those found by Crooks and Sivak [Eqs. (1) and (3) and Refs. [10,11], see Refs. [16,17] for detailed explorations of particular systems], and related bounds for nonequilibrium steady states [18]. Each of these previous bounds roughly correspond to limits in which δS_{bias} is infinite, leading to a deterministic protocol, but where the process leading to that bias is ignored as a source of dissipation. Each of these previous bounds can be made small by lengthening the time or number of steps associated with a protocol. By contrast, neither the time, nor the number of steps enter the bound presented here.

The bounds presented by Crooks and Sivak [10,11] also scale differently with system size. Both the FIM defined in Eq. (4) and the metric introduced in Ref. [10] and Eq. (2) are extensive, scaling linearly with system size in the thermodynamic limit. However, geodesic lengths scale as the square root of the metric. As the geodesic lengths are squared in Eqs. (1) and (3), these bounds are extensive in the precise sense that controlling two identical and uncorrelated systems in tandem requires twice the dissipation. By contrast, the bound presented here scales as the square root of the system size; two identical systems limited by this bound could be controlled in tandem with only $\sqrt{2}$ as much dissipation. Thus, this bound becomes less important in the thermodynamic limit—it is more likely to be relevant for molecular machines than for large engineered systems, where extensive though finite time bounds [10] are more likely to be relevant.

The bound presented here is not in contradiction with the impressive array of experiments verifying the Jarzynski equality and Crooks relations. In these experimental tests, a control parameter is typically manipulated in the nearly deterministic regime. For example, a bead attached to DNA or RNA is manipulated according to a predetermined force protocol by a laser operating as an optical tweezer. The work done by the bead on the nucleotide polymer has been shown to obey both the Jarzynski equality [19] and Crooks relations [20]. It would seem that, as a corollary, these experiments verify the dissipationless adiabatic limit for quasideterministically driven systems. However, in these experimental paradigms the relatively enormous dissipation associated with the laser itself is ignored. In the framework of this Letter, this limit corresponds to the case where δS_{bias} is very large, but where its contribution is ignored.

It remains to be seen whether this bound meaningfully constrains the operation of real biological systems. Biology often utilizes molecular scale motors for which subextensive contributions to energy expenditure could be relevant. In muscle fibers, contraction is initiated by the binding of calcium to myosin, which then hydrolyze ATP to perform mechanical work. Calcium binding acts much as δS_{bias} does here, and biology spends substantial energy maintaining a low intracellular calcium concentration in resting cells. As would be required to approach this bound, many biological motors are able to work in reverse, synthesizing ATP when pulled too hard [21]. However, muscles do not seem to take advantage of this property, hydrolyzing ATP during both extension and contraction under load, and operating far from the reversible regime [22]. Indeed, in many contexts other considerations like speed, reliability, and the constraints of chemistry [23] may matter more than energetic efficiency. England recently used nonequilibrium arguments to estimate the energetic cost of cellular replication, arguing that the dissipation required to rapidly build stable molecules dominates a bacterium's energy budget [24]. Intriguingly, biological systems do seem to seek out highly cooperative, nearly critical states [25], in particular in their cell membrane [26]. These regions of thermodynamic space have, among other properties, a high negative Ricci curvature [13], implying "closeness" to a much larger proper volume of distinct membrane compositions and corresponding physical properties.

Further work will shed light on the extent to which this, or analogous bounds, apply to a larger class of fully realizable systems that do not contain any unphysical, deterministic, elements. This bound would seem to apply to idealized logically reversible computing schemes [27] that have been used to argue against [28] an intrinsic energetic cost associated with the mechanical manipulations underlying computation, leaving only memory erasure as fundamentally requiring dissipation [29]. In addition this bound constrains the efficiency of finite size heat engines. A Carnot cycle operating between temperatures T_h and T_c with pressures P_2 and P_1 bounding the high temperature portion would have to dissipate entropy greater than $\Delta S_{\text{tot}} \geq 4\sqrt{N} \log(P_2/P_1) + 4\sqrt{3N} \log(T_h/T_c)$ in every cycle. Note the $\sqrt{N}$ dependence of this dissipation—in the adiabatic limit an idealized deterministic system would be dissipationless, producing no net entropy but transferring an extensive $Q_h = NT_h \log(P_2/P_1)$ heat out of a hot bath and performing $W = N(T_h - T_c) \log(P_2/P_1)$ work. For fully realizable systems, any directed change must be driven by a finite, though subextensive, production of entropy, even if change is made slowly.

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[1] To be fully symmetric under time reversal, the system must also undergo the application of operators CP, conjugation of charge and inversion of all spatial coordinates. This detail is not thought to contribute meaningfully to the macroscopic breaking of time reversal under discussion here.

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